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The Lubrication of Rollers III. A Theoretical Discussion of Friction and the Temperatures in the Oil Film

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THE LUBRICATION OF ROLLERS

III. A THEORETICAL DISCUSSION OF FRICTION AND THE TEMPERATURES IN THE OIL FILM

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CONTENTS

Measurements of the thickness of the oil film have shown that the lubrication of loaded rollers is hydrodynamic in character. But hitherto the theories of this lubrication have been isothermal; frictional heating has been ignored. For the understanding of the frictional behaviour and also to define the physical conditions met by the oil in its passage through the conjunction of the disks a knowledge of the temperatures in the film is required.

It is from the shearing of the oil in the film that the frictional heat arises. In this paper the dissipation of the heat by conduction through the oil to the surfaces of the disks and by transport with the oil (i.e. convection) are both considered. The temperatures in the film are assessed from the balance between the rates of generation and dissipation of heat.

It is shown that in pure rolling the temperatures arising from friction are confined to the entry side ahead of the region of high pressure and that for the range of conditions considered the temperatures are too low to influence thickness. With sliding it is shown that conduction is the important mechanism of dissipation and that the temperatures due to the sliding rise with pressure. On the entry side where the pressures are low, so are the temperatures. This explains the insensitivity of film thickness to sliding which has been demonstrated experimentally. Within the pressure zone high temperatures occur (e.g. a temperature rise of 200 deg C). The distributions of temperature and viscosity have been found and also the velocity profiles of the oil. These show the cooler oil close to the surfaces of the disks to act almost as a rigid extension of the disks themselves; a situation akin to that postulated in boundary lubrication.

From the viscosity distributions, expressions for the effective viscosity within the pressure zone (i.e. that constant viscosity which would give the same frictional traction) and for the frictional traction have been developed. These display the factors of importance in relation to friction and provide a theoretical background to the experimental measurements of effective viscosity and friction which will be described in part IV.

1. Introduction

It has been shown in parts I (Crook 1958) and II (Crook 1961) of this series of papers that the lubrication of loaded rollers is hydrodynamic in nature and measurements of the thickness of the hydrodynamic film have been described. It has been shown that the thickness of the hydrodynamic film is dependent upon the rate at which oil is drawn into the conjunction of the disks but is insensitive to the variations in the viscosity of the oil which occur once the oil is within the pressure zone. Because of that insensitivity the variations in viscosity were considered only broadly in discussing film thickness. But with respect to friction the variations are of the utmost importance, both in understanding the pattern of frictional behaviour and beyond that in knowing the behaviour of oils under the peculiar physical conditions existing within the pressure zone.

Within that zone the oil experiences pressures dependent upon load and temperatures dependent upon the balance between the generation and the dissipation of the heat arising from the shearing of the oil. Under pressure the viscosities of oils increase; an effect which becomes marked at pressures exceeding 1×10^9 dyn cm^{-2} (Pressure Viscosity Report 1953). But this enhancement of viscosity is reduced by the temperatures due to frictional heating; these temperatures vary across the thickness of the oil film. Consequently across the film the viscosity varies due to differences of temperature whilst along the film the viscosity varies on account of changes both in pressure and in temperature.

In theories of the hydrodynamic lubrication of rollers (Grubin 1949; Dowson & Higginson 1959, 1960; Archard, Gair & Hirst 1961) frictional heating has been neglected. The theories are isothermal, i.e. a viscosity constant across the film is assumed. Here that assumption is not made. The analysis of this paper starts by taking the fundamental hydrodynamic equation in a form consistent with a viscosity varying across the film. From there, general expressions for the velocity gradient in the oil and then for the local rate of generation of heat are derived.

The heat generated must be dissipated by conduction through the oil to the surfaces of the disks and by transport with the oil on its passage through the conjunction, i.e. by convection. For conditions of pure rolling assessments have been made of the regions in which each of these mechanisms is dominant and of the temperatures in the oil due to the rolling friction. The effect upon film thickness of the depression of viscosity due to these temperatures is discussed in relation to a result of part II; there it was shown that experiment gives a film thickness proportional to the half power of the product of the viscosity of the oil at the surface temperature of the disks (η_s) with the mean peripheral speed ($\bar{u}$) whereas theory predicts an exponent of approximately 0·7.

When sliding is introduced the frictional work attributable to the sliding rapidly surpasses that due to the rolling motion. The heat is dissipated predominantly by conduction to the surfaces of the disks (Archard 1959) and the local rate of generation of heat is dependent upon the local viscosity and therefore upon the local temperature. Within the limitations of simplifying assumptions an expression for the temperature distribution has been found. It is shown that in contrast with rolling, when the friction is associated with the entry zone, the friction due to sliding is associated with the pressure zone. This is discussed with relation to the experimental result that of itself sliding has little influence upon film thickness (part II).

From the expression for the temperature distribution temperatures, viscosities and velocity profiles in some typical examples have been calculated. These indicate the physical conditions met by the oil in its passage through the pressure zone. But in addition to such detailed information expressions for the effective viscosity within the pressure zone (i.e. that constant viscosity throughout the zone which would produce the same frictional traction) and for the total frictional traction are derived. These possess an intrinsic interest in that it is important to know the factors affecting friction, but in addition they are required to match the experimental measurements of friction and effective viscosity which will be described in part IV.

2. The fundamental equation

The Navier–Stokes equation after appropriate simplification (Gatcombe 1945) becomes in the co-ordinates of figure 1

$$-\frac{\partial P}{\partial x}=\frac{\partial}{\partial y}\left(\eta\frac{\partial u}{\partial y}\right).$$

When η is taken as being independent of y this reduces to the well-known form

$$-\frac{\partial P}{\partial x}=\eta\frac{\partial^2 u}{\partial y^2},$$

but when considerable heat is dissipated within the oil film the temperature across the film will not be constant and the local viscosity (η) must be treated as a function both of x and y. Then the Navier–Stokes equation has to be taken as

$$-\frac{\partial P}{\partial x}=\eta\frac{\partial^2 u}{\partial y^2}+\frac{\partial \eta}{\partial y}\frac{\partial u}{\partial y},$$

where P is the pressure and u is the velocity of the oil in the negative direction of x.

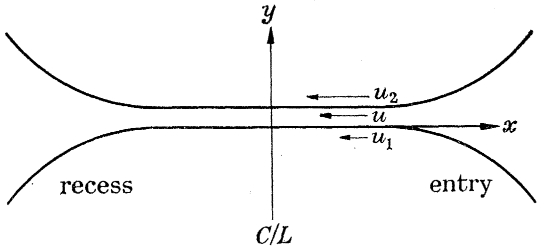

Figure 1. Co-ordinates.

Two successive integrations with respect to y and the boundary conditions

$$y=0,\quad u=u_1,\quad \eta=\eta_1;\qquad y=h,\quad u=u_2,\quad \eta=\eta_2$$

give
$$\frac{\partial u}{\partial y}=-\frac{1}{\eta}\frac{\partial P}{\partial x}\left[y-\frac{h}{2}\right]+\frac{\eta_2 u_2}{\eta h}-\frac{\eta_1 u_1}{\eta h}-\frac{1}{\eta h}\int_0^h u\frac{\partial \eta}{\partial y}\,dy, \tag{2·1}$$

where h is the thickness of the film at x. If the variation of u with y be continuous, somewhere

$$\frac{\partial u}{\partial y} = \left(\frac{u_2 - u_1}{h}\right).$$

At one place where that occurs let

$$y = h/m, \quad \eta = \eta_m$$

and then it may be shown from equation (2·1) that

$$\frac{\partial u}{\partial y} = -\frac{1}{\eta}\frac{\partial P}{\partial x}\left[y - \frac{h}{m}\right] + \frac{\eta_m}{\eta}\frac{(u_2 - u_1)}{h}. \tag{2·2}$$

3. The heat balance

(a) *The generation of heat*

The rate of generation of heat per unit volume (q) is given by the product of stress with the rate of strain, i.e. by

$$q = \eta(\partial u/\partial y)^2. \tag{3·1}$$

(b) *The dissipation of heat*

It will be assumed that heat is dissipated according to the equation (Carslaw & Jaeger 1947; Hunter & Zienkiewicz 1960)

$$\rho c u \frac{\partial \theta}{\partial x} + K\frac{\partial^2 \theta}{\partial y^2} = -q, \tag{3·2}$$

where θ denotes temperature, ρ density, c specific heat and K thermal conductivity all with respect to the oil. The first term on the left relates to the transport of heat by the oil and will be referred to as convection. The second term relates to conduction of heat across the film. Conduction in the x direction is ignored because as $h \ll (x_2 - x_1)$ the temperature gradients in the x direction must be small compared with those across the film.

It has been found too difficult to retain both terms on the left simultaneously in the present analysis. It will, however, be argued that in general the conduction term is predominant.

4. The oil temperatures in pure rolling

When u_1 and u_2 are equal the velocity profiles must be symmetrical about the median plane and it follows from equation (2·2) that

$$\frac{\partial u}{\partial y} = -\frac{1}{\eta}\left(\frac{\partial P}{\partial x}\right)y', \quad y' = y - \tfrac{1}{2}h.$$

Therefore (equation (3·1))

$$q = \frac{1}{\eta}\left(\frac{\partial P}{\partial x}\right)^2 y'^2. \tag{4·1}$$

(a) *Conduction*

If conduction were the only mechanism of dissipation then

$$K\frac{\partial^2 \theta}{\partial y^2} = -\frac{1}{\eta}\left(\frac{\partial P}{\partial x}\right)^2 y'^2. \tag{4·2}$$

The local viscosity (η) will vary with temperature and pressure. If θ_s be the surface temperature on the entry side the viscosity of the oil as it enters the conjunction will be η_s

(part II). Within the conjunction it will be assumed that the surface temperature remains θ_s and that the oil viscosity at the surfaces (η_x) which will differ from η_s because of the rise in pressure, is given by

$$\eta_x = \eta_s \exp(\delta P).$$

Away from the surfaces of the disks, because of the heating of the oil due to friction, the temperature will exceed θ_s. The excess will be denoted by θ rising to θ_c on the median plane of the film. It will be assumed that the local viscosity η is given by

$$\eta = \eta_x \exp(-\gamma\theta). \tag{4·3}$$

It follows that

$$\eta = \eta_s \exp(\delta P - \gamma\theta).$$

However, if η be so expressed it leads to an equation for which the author has found no analytical solution. But if η be expressed as

$$\eta = \eta_s \exp(\delta P - \gamma\theta_c)$$

the solution will be an over-estimate of θ because η has been given too low a value and by equation (4·1) the value of q has thereby been exaggerated.

If η be so expressed, from equation (4·2)

$$\partial^2\theta/\partial y'^2 = -By'^2 \exp(\gamma\theta_c),$$

where

$$B = \left(\frac{\partial P}{\partial x}\right)^2 \frac{\exp(-\delta P)}{K\eta_s}.$$

With the boundary conditions

$$y' = 0, \quad \partial\theta/\partial y = 0; \qquad y' = \tfrac{1}{2}h, \quad \theta = 0,$$

this gives

$$\theta_c \approx Bh^4 \exp(\gamma\theta_c)/192.$$

It is more convenient to recast B so that the evaluation of small pressure gradients is avoided. For this the well-known isothermal expression

$$-\frac{\partial P}{\partial x}\exp(-\delta P) = 12\bar{u}\eta_s\left(\frac{h-h_D^*}{h^3}\right), \tag{4·4}$$

where h_D^* is the film thickness at the pressure maximum, was used and then

$$\theta_c \exp(-\gamma\theta_c) \approx \frac{3(\bar{u})^2\eta_s\alpha^2\exp(\delta P)}{4K}, \quad \alpha = (1-h_D^*/h). \tag{4·5}$$

Both α and $\exp(\delta P)$ tend to unity far from the pressure zone so there the value of θ_c given by equation (4·5) rises to the limit

$$\lim_{x\to\infty} \theta_c \exp(-\gamma\theta_c) = 3(\bar{u})^2\eta_s/4K. \tag{4·6}$$

(b) *Convection*

A general solution assuming convection to be the only mechanism of dissipation has not been found but from the equation

$$\rho cu\left(\frac{\partial\theta}{\partial x}\right) = -q = -\frac{1}{\eta}\left(\frac{\partial P}{\partial x}\right)^2 y'^2,$$

it may be shown that on the entry side where $h \gg h_D^*$, close to the surfaces of the disks

$$\theta \approx 3P/\rho c. \tag{4·7}$$

(c) *The regions of conduction and of convection*

The curves of pressure and of h given in figure 2 (Archard *et al.* 1961) were used to estimate θ_c as given by equation (4·5). (γ was taken as 2×10^{-2} deg C^{-1} and δ as $1{\cdot}6 \times 10^{-9}$ dyn^{-1} cm^2. These values will be used throughout this paper.) The estimate is displayed as curve (*a*) of figure 3; curve (*b*) is a similar estimate for a lower rolling speed for which curves similar to those of figure 2 were available. Clearly the temperatures required to dissipate the heat by conduction across the film are so small that for them exp $(\gamma\theta_c)$ may be taken as unity. This justifies the use of an isothermal expression for $\partial P/\partial x$ (equation (4·4)).

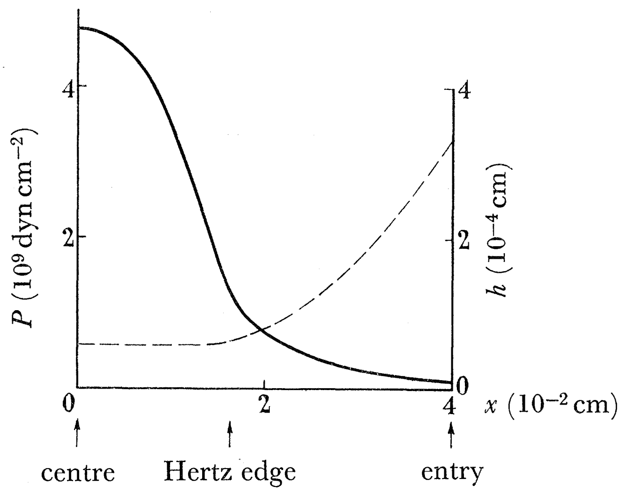

FIGURE 2. The pressure distribution (P) and gap (h) on the entry side for steel disks of 7·6 cm diameter. Load, $1{\cdot}2 \times 10^8$ dyn cm^{-1} (680 Lb. in.$^{-1}$); $\bar{u}$, 230 cm s^{-1} (7·4 ft. s^{-1}); η_s, 0·4 P; δ, $1{\cdot}6 \times 10^{-9}$ dyn^{-1} cm^2. ——, P; ----, h. (Curves due to Archard *et al.* 1961.)

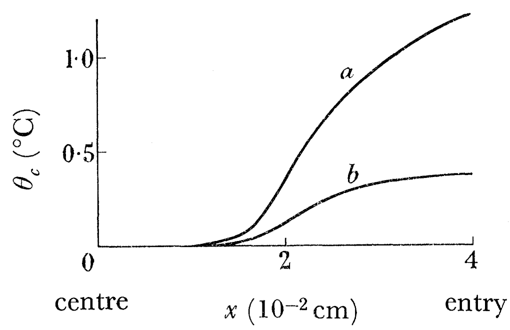

FIGURE 3. The temperature rise on the median plane (θ_c) for pure rolling when conduction alone is considered. (*a*) $\bar{u}$, 230 cm s^{-1} (7·4 ft. s^{-1}); (*b*) $\bar{u}$, 130 cm s^{-1} (4·2 ft. s^{-1}). Thermal conductivity of oil taken as 1×10^4 dyn s^{-1} deg C^{-1}; γ, 2×10^{-2} deg C^{-1}; data otherwise as for figure 2.

Far from the pressure zone the values of θ_c given by equation (4·5) rise to the limit given by equation (4·6) but there, because of the vanishing pressure, the value of θ given by equation (4·7) falls to zero. The dominant mechanism of heat dissipation will be that which operating alone would require the lesser temperature. Consequently, far from the pressure zone convection is the dominant mechanism but as the pressure rises so does the temperature given by equation (4·7) until, for instance, at a pressure of 1×10^7 dyn cm^{-2} it gives a value of θ (1·6 deg C; ρc taken as 2×10^7 erg cm^{-3} deg C^{-1}) equal to the limiting value of θ_c for curve (*a*) of figure 3. Consequently there is a region in which convection and conduction will both be effective and in which the temperature rises to a maximum less than the limiting value of θ_c. In this instance that region will occur where the pressures are of the order 1×10^7 dyn cm^{-2}. Reference to figure 2 shows that such pressures are far from the pressure zone. As the pressure zone is approached more closely, conduction requires the lower temperature (figure 3), the temperature in fact vanishing within the pressure zone when h, as in figure 2, becomes constant (α of equation (4·5) becomes zero). Thus, except for a region far removed from the pressure zone, convection plays little part in the dissipation of the frictional heat and the temperatures in pure rolling are effectively those needed to dissipate the heat by conduction across the thickness of the film.

(*d*) *Film thickness*

The question can now be discussed whether the difference between the relationships of film thickness to $\bar{u}\eta_s$ as found by experiment and as predicted by theory, are resolvable in terms of the heating of the oil by friction. (Pure rolling alone will be considered.) Experiment showed that $h_D^* \propto (\bar{u}\eta_s)^{0\cdot5}$, whereas theory predicts the exponent of $\bar{u}\eta_s$ to be approximately 0·7 (part II). In the appendix of part II it is shown that, to resolve the discrepancy in terms of surface temperature, the temperatures as measured would have had to be progressively too low by 14 deg C for each decade by which $\bar{u}\eta_s$ increases. The issue now becomes whether the discrepancy can be resolved in terms of the oil temperature being raised above the surface temperature by frictional heat.

If η_s be taken as 0·4 P the limiting value of θ_c (equation (4·6)) for $(\bar{u}\eta_s)$ equal to 150 dyn cm^{-1} is, for instance, only 4·2 deg C. Consequently any decade interval in $(\bar{u}\eta_s)$ with an upper value below 150 dyn cm^{-1} could not possibly change the oil temperature by the 14 deg C required to account for the discrepancy. At higher values of $(\bar{u}\eta_s)$ the limiting values of θ_c are greater. But because of the moderating influence of convection the limiting value of θ_c will nowhere be attained and, furthermore, because film thickness depends upon viscosity throughout the entry zone, the representative temperature with respect to thickness will be less than the maximum temperature within that zone. Indeed θ_c itself falls to zero as the pressure zone is approached and it relates only to the median plane. Away from the median plane the temperatures must fall. In fact experiment has shown that up to the highest value of $(\bar{u}\eta_s)$ explored experimentally (480 dyn cm^{-1}) proportionality with $(\bar{u}\eta_s)^{0\cdot5}$ was maintained (part II) which suggests that even up to that value of $(\bar{u}\eta_s)$ the temperatures due to rolling friction were not influencing the film thickness.

5. The oil temperatures due to sliding

It will be assumed that when sliding is introduced the term in $\partial P/\partial x$ of equation (2·2) may be neglected. (This assumption is discussed in appendix A.)

The neglect of the term in $\partial P/\partial x$ is equivalent to disregarding the rolling component of the frictional traction for if T be the total frictional traction per unit face width then, for instance, from equation (2·2)

$$\left(\frac{\partial T}{\partial x}\right)_{y=0} = \left(\eta\frac{\partial u}{\partial y}\right)_{y=0} = \frac{\partial P}{\partial x}\frac{h}{m} + \frac{\eta_m(u_2-u_1)}{h}.$$

When the disks roll the term in $\partial P/\partial x$ alone remains and therefore it can be considered to represent the rolling component of the friction.

When the term in $\partial P/\partial x$ is neglected equation (2·2) becomes

$$\frac{\partial u}{\partial y} = \frac{\eta_m}{\eta}\frac{(u_2-u_1)}{h} \tag{5·1}$$

and therefore
$$\int_0^h \frac{\partial u}{\partial y}\,dy = (u_2-u_1) = \frac{\eta_m(u_2-u_1)}{h}\int_0^h \frac{dy}{\eta}.$$

It follows that
$$\frac{1}{\eta_m} = \frac{1}{h}\int_0^h \frac{dy}{\eta} \tag{5·2}$$

which defines η_m in terms of the local viscosity η.

(a) Conduction

From equations (3·1) and (5·1)
$$q = \frac{\eta_m^2}{\eta}\frac{(u_2-u_1)^2}{h^2}. \tag{5·3}$$

In a discussion of the temperature of rubbing surfaces Archard (1959) considered the dissipation of heat within the oil film. For simplicity he assumed q to be constant and concluded that conduction across the thickness of the film was dominant. It will also be assumed here that conduction is dominant; the assumption is discussed in appendix B.

If the dissipation of heat were by conduction alone then

$$K\frac{\partial^2\theta}{\partial y^2} = \frac{-\eta_m^2}{\eta}\frac{(u_2-u_1)^2}{h^2}. \tag{5·4}$$

Integration with respect to y together with equation (5·2) gives

$$\left[\frac{\partial\theta}{\partial y}\right]_0^{y=h} = \frac{-\eta_m}{K}\frac{(u_2-u_1)^2}{h}. \tag{5·5}$$

If
$$\eta = \eta_x \exp(-\gamma\theta),$$

equation (5·4) can be written as

$$\frac{\partial^2\theta}{\partial y^2} = -A\exp(\gamma\theta), \quad A = \frac{\eta_m^2}{K\eta_x}\frac{(u_2-u_1)^2}{h^2}$$

and integration with respect to y gives

$$\frac{\partial\theta}{\partial y} = \pm\left[2\omega\left(1-\frac{A}{\gamma\omega}\exp(\gamma\theta)\right)\right]^{\frac{1}{2}}, \tag{5·6}$$

where ω is a constant of integration.

Equation (5·6) suggests that θ is symmetrical about the median plane where it rises to a maximum given by
$$\exp(\gamma\theta_c) = \gamma\omega/A. \tag{5·7}$$

From a comparison of the expressions for $\partial\theta/\partial y_{0,h}$ given by equations (5·5) and (5·6) ω may be evaluated. By symmetry

$$\left(\frac{\partial\theta}{\partial y}\right)_{y=0} = -\left(\frac{\partial\theta}{\partial y}\right)_{y=h}$$

and therefore from equation (5·5)

$$\left(\frac{\partial\theta}{\partial y}\right)_{y=0} = -\left(\frac{\partial\theta}{\partial y}\right)_{y=h} = \frac{\eta_m}{2K}\frac{(u_2-u_1)^2}{h}.$$

Another expression for $(\partial\theta/\partial y)_0$ is obtained by setting θ to zero in equation (5·6). Then

$$\left(\frac{\partial\theta}{\partial y_0}\right)^2 = \left(\frac{\partial\theta}{\partial y_h}\right)^2 = 2\omega\left(1-\frac{A}{\gamma\omega}\right).$$

From a comparison of the expressions for $(\partial\theta/\partial y_{0,h})^2$ arising from the above two equations it may be shown that

$$\omega = \frac{8\eta_m^2\psi(\psi+1)}{\eta_x^2h^2\psi^2}, \tag{5·8}$$

where

$$\psi = \frac{\eta_x\gamma(u_2-u_1)^2}{8K}, \tag{5·9}$$

and that consequently (equation (5·7))

$$\exp(\gamma\theta_c) = (\psi+1) = \left(\frac{\eta_x\gamma(u_2-u_1)^2}{8K}+1\right). \tag{5·10}$$

When $\psi \gg 1$

$$\theta_c \approx \frac{\delta P}{\gamma}+\frac{1}{\gamma}\left[\ln\left(\frac{\eta_s\gamma}{8K}\right)+2\ln(u_2-u_1)\right].$$

It follows from equation (5·10) that θ_c depends upon both η_s and load; it is independent of $\bar{u}$ so if η_s be constant it is then also independent of film thickness.

(b) *The variations of θ and η across the film*

Integration of equation (5·6) with respect to y with the boundary condition

$$y = 0, \quad \theta = 0$$

gives

$$\ln\left[\frac{1-\surd\{1-\exp(\gamma\theta)/\exp(\gamma\theta_c)\}}{1+\surd\{1-\exp(\gamma\theta)/\exp(\gamma\theta_c)\}}\right]-\ln\left[\frac{1-\surd\{1-\exp(-\gamma\theta_c)\}}{1+\surd\{1-\exp(-\gamma\theta_c)\}}\right] = y\gamma\surd(2\omega). \tag{5·11}$$

When y is $\frac{1}{2}h$, $\theta = \theta_c$ and the first logarithmic term on the left is zero so

$$\omega = \frac{2}{h^2\gamma^2}\left\{\ln\left[\frac{1-\surd\{1-\exp(-\gamma\theta_c)\}}{1+\surd\{1-\exp(-\gamma\theta_c)\}}\right]\right\}^2 \tag{5·12}$$

and equation (5·11) may be written as ($\surd\omega$ taken positive)

$$\left[1-\frac{2y}{h}\right]\ln\left[\frac{1-\surd(1-\eta_c/\eta_x)}{1+\surd(1-\eta_c/\eta_x)}\right] = \ln\left[\frac{1-\surd(1-\eta_c/\eta)}{1+\surd(1-\eta_c/\eta)}\right], \tag{5·13}$$

where the temperatures are now expressed in terms of viscosity.

When $\eta = \eta_m$, $y = h/m$. Consequently from the values of y which satisfy equation (5·13) when η_m is substituted for η, values of m may be found (appendix C).

(c) *Numerical values*

Values of θ_c calculated from equation (5·10) are shown as a function of pressure for three sliding speeds in figure 4 (*a*). (As before, it was assumed that η_x is given by $\eta_s \exp(\delta P)$.) The temperatures rise rapidly with pressure.

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In figure 4 (*b*) the values of $\exp(-\gamma\theta_c)$ at a pressure of $1\times10^9\,\mathrm{dyn\,cm^{-2}}$ have been plotted against (u_2-u_1). This figure will be discussed in § 6.

The curves of figure 4 (*a*) were used to obtain the values of θ_c corresponding to the pressure curve of figure 2. The results are given in figure 5.

From equation (5·13) the curves of figure 6, in which relative viscosity is plotted as a function of y/h, were obtained (logarithmic-linear plot). From these curves the variation

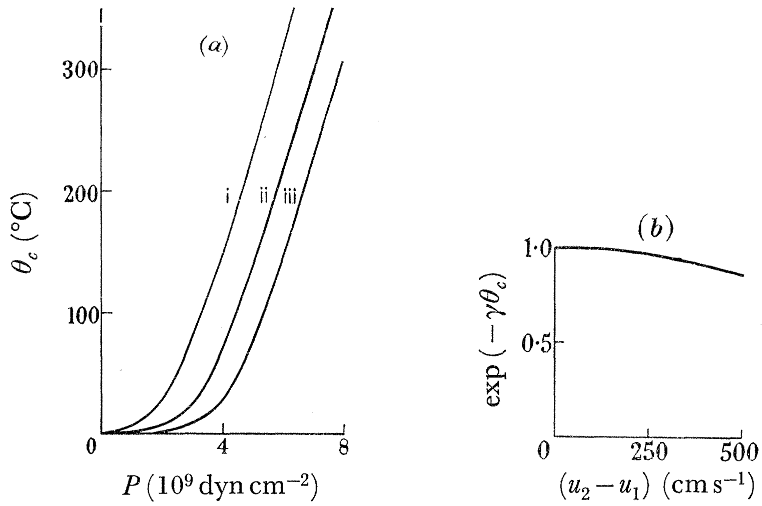

FIGURE 4. The temperature rise on the median plane (θ_c) due to sliding. Conduction alone considered. (*a*) As a function of pressure. Sliding speeds: (i) 500 cm s^{-1}, (ii) 200 cm s^{-1}, (iii) 100 cm s^{-1}. (*b*) Exp $(-\gamma\theta_c)$ at a pressure of 1×10^9 dyn cm^{-2} as a function of sliding speed.

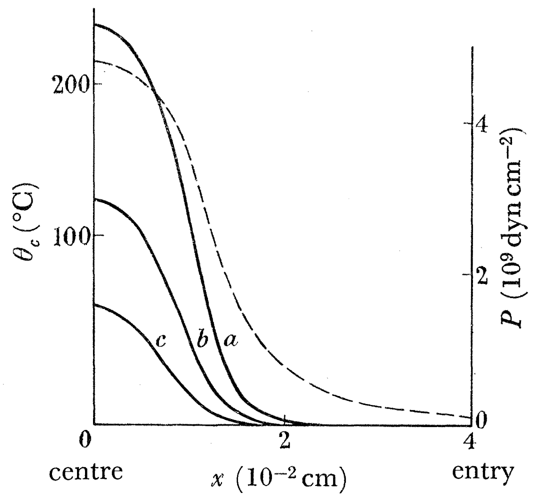

FIGURE 5

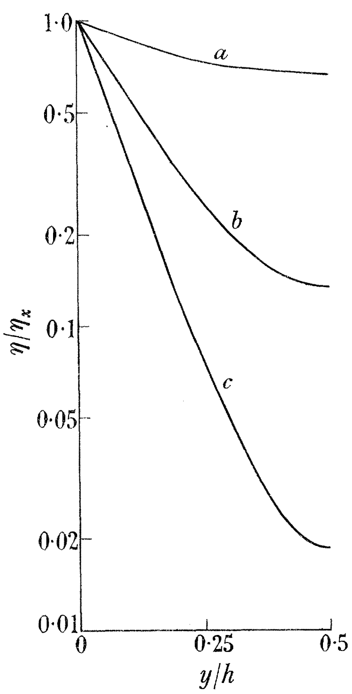

FIGURE 6

FIGURE 5. The temperature rise on the median plane (θ_c) as a function of x. ——, θ_c; – – – –, pressure (same as figure 2). Sliding speeds: (*a*) 500 cm s^{-1}, (*b*) 200 cm s^{-1}, (*c*) 100 cm s^{-1}.

FIGURE 6. η/η_x as a function of y/h for three values of θ_c: (*a*) 20 deg C, (*b*) 100 deg C, (*c*) 200 deg C.

of θ with y was deduced and the results are given in figure 7. In addition relative values of $\partial u/\partial y$ were calculated (equation (5·1)) and by numerical integration the velocity profiles of figure 8 were obtained; the greater the temperature rise the more does the shear become concentrated about the mid-point. Thus when θ_c is 200 deg C the curve shows that 80 % of the slip occurs in a central band 0·4 h wide.

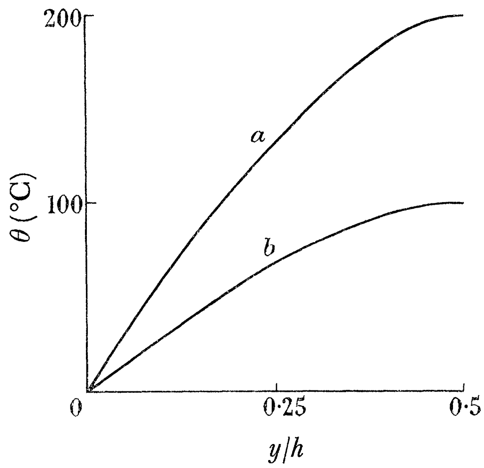

FIGURE 7

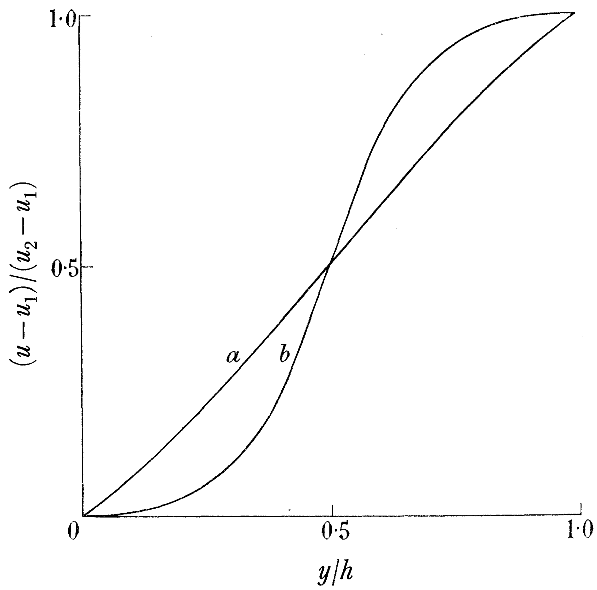

FIGURE 8

FIGURE 7. Oil temperature rise (θ) as a function of y/h for two values of θ_c: (a) 200 deg C, (b) 100 deg C.
FIGURE 8. Velocity profiles for two values of θ_c: (a) 10 deg C, (b) 200 deg C.

From equation (5·10) the sliding speed at which the effect of the temperature due to frictional heat upon viscosity just offsets the effect of pressure may be estimated. When $\psi \gg 1$ equation (5·10) may be written as

$$\exp(\gamma\theta_c - \delta P) = \eta_s \gamma (u_2 - u_1)^2/8K.$$

The criterion is that the exponential should be unity and this occurs when (u_2-u_1) is approximately 3000 cm s^{-1} ($\eta_s = 0·4$ P). This is greatly in excess of the sliding speeds employed in the experiments of parts II and IV (up to 480 cm s^{-1}).

6. SLIDING AND FILM THICKNESS

It has been shown in § 4 (a) of part II that over the greater part of the pressure zone the film thickness must be effectively constant and also that the value of that constant thickness is determined by the quantity of oil per unit time reaching the section of the film where the pressure reaches 1×10^9 dyn cm^{-2}. From integration of the appropriate form of the Navier–Stokes equation it is well known that the quantity of oil and therefore the film thickness depends upon viscosity and (u_2+u_1) but not explicitly upon (u_2-u_1). Consequently at constant (u_2+u_1) the film thickness will only vary with (u_2-u_1) if sliding affects the viscosity of the oil on the entry up to the section where the pressure is 1×10^9 dyn cm^{-2}.

In figure 4 (b) the values of $\exp(-\gamma\theta_c)$ as obtained from equation (5·10) when the pressure is 1×10^9 dyn cm^{-2} are plotted as a function of sliding speed. The curve represents the relative reduction in the viscosity at the mid-point of the film due to the heat of sliding. The introduction of 500 cm s^{-1} sliding produces a fall of 12 %. But because a pressure of 1×10^9 dyn cm^{-2} occurs where convection is important (appendix B) and because in figure 4 (b) convection is neglected, the reduction in viscosity must be less than the figure suggests. Furthermore, away from the mid-point and also at lower pressures the reduction must be even smaller.

Consequently up to the section where the pressure is 1×10^9 dyn cm^{-2} the heat of sliding has little effect upon viscosity. Theory therefore supports the suggestion made previously (part I) that the film thickness is determined by conditions on the entry side ahead of the region in which the viscous forces and heating are intense and also the theory is consistent with the experimental observation (part II) that the introduction of sliding has little effect upon film thickness.

7. The effective viscosity

The effective viscosity within the pressure zone is defined as that constant viscosity throughout the zone which would give the same frictional traction as the system actually exhibits.

The relation of the effective viscosity ($\bar{\eta}_m$) to η_m follows from equation (5·1). As the shear stress is given by $\eta\, \partial u/\partial y$ it is evident from that equation that the stress is independent of y. Consequently it has the same value at the surfaces of the disks as elsewhere, and it follows (when the small frictions at entry and recess are disregarded) that the frictional traction T is given by

$$T = \frac{(u_2 - u_1)}{h_D^*} \int_{x_1}^{x_2} \eta_m \, \mathrm{d}x, \tag{7·1}$$

where $x_{1,2}$ are the limits of the pressure zone and over which it is assumed that h is effectively constant and has the value h_D^* (§ 4 (a) of part II).

If h_D^* be constant the conjunction may be regarded as a parallel plate viscometer. It then follows that T is also given by

$$T = \bar{\eta}_m (u_2 - u_1)\,(x_2 - x_1)/h_D^*, \tag{7·2}$$

where $\bar{\eta}_m$ is the effective viscosity. Consequently (equations (7·1) and (7·2))

$$\bar{\eta}_m = \frac{1}{(x_2 - x_1)} \int_{x_1}^{x_2} \eta_m \, \mathrm{d}x. \tag{7·3}$$

This defines $\bar{\eta}_m$ as a function of η_m which itself will now be evaluated.

In terms of ψ equation (5·12) can be written as

$$\sqrt{\omega} = \frac{\sqrt{2}}{h\gamma} \ln \left[\frac{\sqrt{(\psi+1)} + \sqrt{\psi}}{\sqrt{(\psi+1)} - \sqrt{\psi}} \right].$$

From a comparison of this with equation (5·8) it follows that

$$\eta_m = \eta_x f(\psi), \quad f(\psi) = \frac{\ln \{\sqrt{(\psi+1)} + \sqrt{\psi}\}}{\sqrt{\{\psi(\psi+1)\}}}. \tag{7·4}$$

A graph of $f(\psi)$ is given in figure 9. By use of this the values of η_m given in figure 10 were obtained.

When ψ is large compared with unity

$$f(\psi) \approx \ln(4\psi)/2\psi. \tag{7·5}$$

The interrupted line in figure 9 presents this approximation.

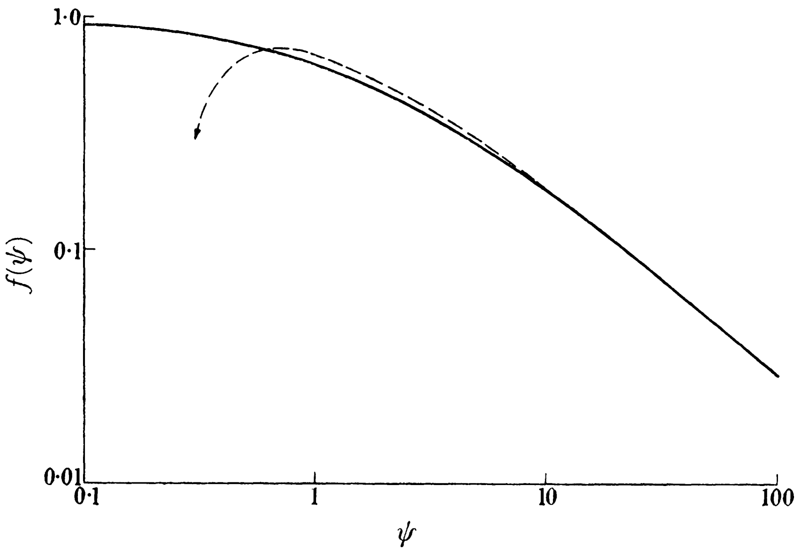

FIGURE 9. $f(\psi)$ as a function of ψ. ——, Equation (7·4); ----, equation (7·5).

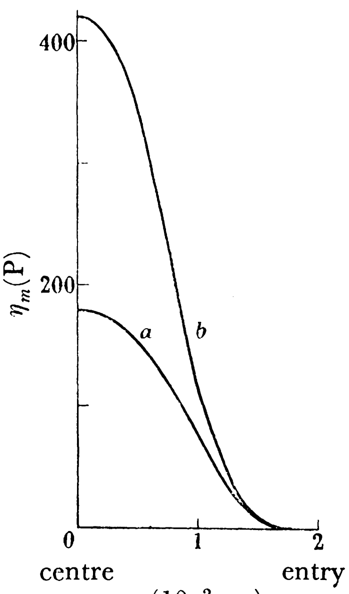

FIGURE 10

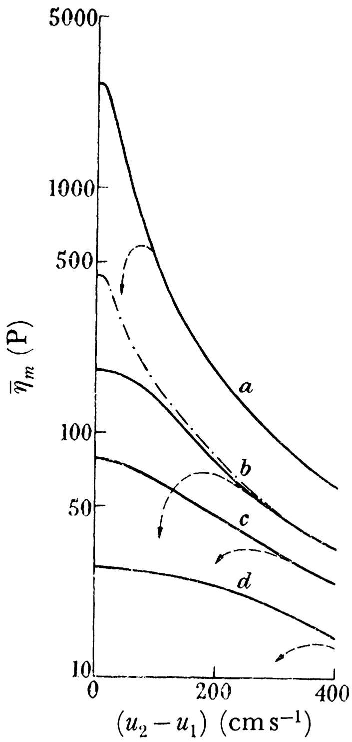

FIGURE 11

FIGURE 10. η_m as a function of x for the pressure curve of figure 2. (a) Sliding speed 200 cm s^{-1}; (b) sliding speed 100 cm s^{-1}.

FIGURE 11. The effective viscosity ($\bar{\eta}_m$) as a function of sliding speed. Load: (a) 2×10^8 dyn cm^{-1}, (b) 1×10^8 dyn cm^{-1}, (c) $7·5 \times 10^7$ dyn cm^{-1}, (d) 5×10^7 dyn cm^{-1}. ——, $\eta_s = 0·4$ P; –·–·–, $\eta_s = 1·0$ P (both by numerical integration); ——, $\eta_s = 0·4$ P, from equation (7·7).

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The effective viscosity may be found by numerically integrating curves of η_m such as those of figure 10 over the full pressure zone (equation (7·3)). However, it is usually sufficient, in calculating values of η_m, to assume a Hertzian pressure distribution. The Hertzian distribution differs appreciably from the true hydrodynamic distribution only where the pressures are low. Thus the major errors occur where the values of η_m are on any reckoning low and it may be judged from figure 10 that such errors will little influence the value of $\bar{\eta}_m$. Calculations of $\bar{\eta}_m$ based upon a Hertzian distribution are given in figure 11. In part IV it will be shown that the experimental curves are similar.

When it can be taken that over the significant part of the pressure zone ψ is so large that the approximation of equation (7·5) is applicable, numerical integration is unnecessary, for then from that equation

$$\eta_m = \frac{8K}{\gamma(u_2-u_1)^2}\left[\tfrac{1}{2}\delta P + \ln(u_2-u_1) + \tfrac{1}{2}\ln\left(\frac{\eta_s\gamma}{2K}\right)\right], \tag{7·6}$$

and, on the assumption that the pressure zone has the Hertzian width $2b$, integration with respect to x gives

$$\bar{\eta}_m = \frac{4K}{b\gamma(u_2-u_1)^2}\left[\tfrac{1}{2}\delta F + 2b\ln(u_2-u_1) + b\ln\left(\frac{\eta_s\gamma}{2K}\right)\right], \tag{7·7}$$

where F is the load per unit face width. The interrupted lines in figure 11 were calculated from this approximation. They show its range of applicability.

(Equation (7·6) is implicit in an expression for the local frictional traction quoted by Petrusevich (1951) without derivation or discussion of its physical basis. Petrusevich's expression incorporates his own expression for film thickness. The dotted lines of figure 11 show that equation (7·6) leads to serious errors at low sliding speeds. At the rolling point there can be no frictional heat to modify the effect of pressure upon viscosity and there the effective viscosity must reach its greatest value. The approximation does not satisfy this requirement.)

Equation (7·7) suggests that in general $\bar{\eta}_m$ is affected by changes in η_s far less than proportionately. This is demonstrated by comparing the solid and chain dotted curves (b) of figure 11. Both were calculated by numerical integration; the solid curve for an η_s of 0·4 P and the chain-dotted curve for an η_s of 1 P. At zero slip $\bar{\eta}_m$ is of course proportional to η_s but at a slip of 400 cm s^{-1} the difference between the curves has disappeared.

8. The frictional traction

The tractions of figure 12 were calculated from the values of $\bar{\eta}_m$ given in figure 11 and from the expression

$$T = 2\bar{\eta}_m b(u_2-u_1)/h_D^*, \tag{8·1}$$

which is the same as equation (7·2) except that (x_2-x_1) has been replaced by the Hertzian width $2b$. In addition, in calculating the tractions the assumption has been made that $(\bar{u})$ remained constant as (u_2-u_1) was varied. Because $h_D^* \propto [\bar{u}\eta_s]^{0·5}$ this assumption allowed h_D^* to be regarded as constant for constant η_s. In fact h_D^* was taken as 1 μ for an η_s of 0·4 P and as $\sqrt{(1/0·4)}\,\mu$ (1·58 μ) for an η_s of 1 P.

Curve a of figure 12 (load 2×10^8 dyn cm^{-1}) exhibits a maximum as do the others to a lesser degree. The obvious explanation of the maximum is that at small slips there is little frictional heat to affect viscosity and consequently the traction rises proportionately with slip (equation (8·1)), while at high slips the reduction in viscosity due to frictional heat can become the dominant influence and, as in curve *a* in particular, eventually the traction may fall as the slip increases.

An interesting consequence of the tendency of $\bar{\eta}_m$ to become independent of η_s at high slips (cf. the solid and chain-dotted curves (*b*) of figure 11 for values of η_s of 0·4 and 1 P respectively) is that the friction falls as η_s is increased (cf. curves *b* and *c* of figure 12). This is because as η_s is increased h will increase, but $\bar{\eta}_m$ will remain constant and it follows from equation (8·1) that T will then fall. Indeed a fall in friction as η_s is increased has been reported by Misharin (1958) and has been observed by the author (part IV).

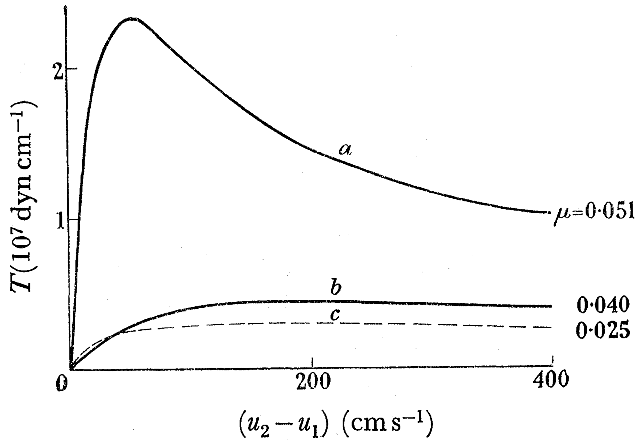


FIGURE 12. Frictional traction (T) as a function of slip (u_2-u_1) when $\bar{u}$ constant. (*a*) Load, 2×10^8 dyn cm^{-1}; η_s, 0·4 P. (*b*) Load, 1×10^8 dyn cm^{-1}; η_s, 0·4 P. (*c*) Load, 1×10^8 dyn cm^{-1}; η_s, 1 P. μ, Coefficient of friction.

If η_s be constant, h_D^* is proportional to $(\bar{u})^{0\cdot5}$. If in addition (u_2-u_1) is held constant whilst $\bar{u}$ varies, $\bar{\eta}_m$ will be constant because ψ (equation (5·9)) is independent of $\bar{u}$. It then follows that for these conditions T should be proportional to $(\bar{u})^{-0\cdot5}$. It will be shown in part IV that T does fall as $\bar{u}$ increases.

Lastly the theory does predict coefficients of friction (which are indicated in figure 12) of the same order as those found experimentally (e.g. Misharin 1958).

9. The surface temperatures within the pressure zone

It has been assumed in the previous calculations that the temperature of the surfaces is constant. It was assumed in particular that the viscosity at the surfaces (η_x) is given by

$$\eta_x = \eta_s \exp \delta P.$$

But the heat generated within the film passes into the surfaces and raises their temperature, so more properly

$$\eta_x = \eta_s \exp(\delta P - \gamma\theta_x), \tag{9·1}$$

where θ_x is the surface temperature at the point x. (The rise in surface temperature due to the friction at solid-solid contacts has been discussed by Blok (1937).)

If it is assumed that the heat flows equally to both surfaces then the rate at which it passes into one surface at any value of x is given by

$$Q = \frac{1}{2}\int_0^h q\,\mathrm{d}y$$

which, by equations (5·2) and (5·3), may be written as

$$Q = (\eta_m/2h)\,(u_2-u_1)^2. \tag{9·2}$$

Thus if h be constant the distribution of heat flow into the surfaces is the same as that of η_m (figure 10).

At a time τ after the supply to a surface of a quantity of heat per unit area $Q\,\mathrm{d}\tau$ the temperature rise is given by (Carslaw & Jaeger 1947)

$$\mathrm{d}\theta = \frac{1}{(\pi K\rho c)^{\frac{1}{2}}}\frac{Q\,\mathrm{d}\tau}{\tau^{\frac{1}{2}}}, \tag{9·3}$$

where K, ρ and c are respectively the thermal conductivity, density and specific heat of the material of the disks. Thus in the present context for the disk of peripheral speed u_1

$$\theta_{(x=a)} = \frac{1}{(\pi K\rho c)^{\frac{1}{2}}}\frac{1}{u_1^{\frac{1}{2}}}\int_0^a \frac{Q\,\mathrm{d}x}{(a-x)^{\frac{1}{2}}}, \tag{9·4}$$

where x is now measured from the entry edge of the pressure zone into the pressure zone.

The estimation of θ from equations (9·2) and (9·4) is simplified by the fact that θ does not depend sensitively upon the distribution of Q. For instance, it may be shown that a parabolic distribution gives a maximum temperature

$$\frac{2\cdot08\,Q_{\text{av.}}}{(\pi K\rho c)^{\frac{1}{2}}}\left(\frac{2b}{u_1}\right)^{\frac{1}{2}}$$

at

$$x = 1\cdot5b,$$

while a distribution in the form of an isosceles triangle gives a maximum temperature

$$\frac{2\cdot18\,Q_{\text{av.}}}{(\pi K\rho c)^{\frac{1}{2}}}\left(\frac{2b}{u_1}\right)^{\frac{1}{2}}$$

at

$$x = 1\cdot33b,$$

where $Q_{\text{av.}}$ is the average value of Q and $2b$ is the base of the distributions.

It is more convenient to continue the argument with reference to a particular example. Conditions corresponding to curve a of figure 11 at a slip of $400\,\mathrm{cm\,s^{-1}}$ will be chosen. Curve (i) of figure 13 (a) gives the distribution of η_m. Clearly it is close to parabolic so a parabolic distribution of Q will be assumed. Integration of equation (9·4) then gives

$$\theta_x = \frac{0\cdot8\,Q_{\text{av.}}}{(\pi K\rho c)^{\frac{1}{2}}}\left(\frac{b}{u_{1,2}}\right)^{\frac{1}{2}} f(\beta), \quad \beta = \frac{x}{b}, \quad f(\beta) = \beta^{\frac{1}{2}}(5\beta-2\beta^2).$$

When $Q_{\text{av.}}$ is expressed in terms of the frictional traction T this becomes

$$\theta_x = \frac{0\cdot2\,T(u_2-u_1)}{(\pi K\rho c)^{\frac{1}{2}}\,(bu_{1,2})^{\frac{1}{2}}} f(\beta). \tag{9·5}$$

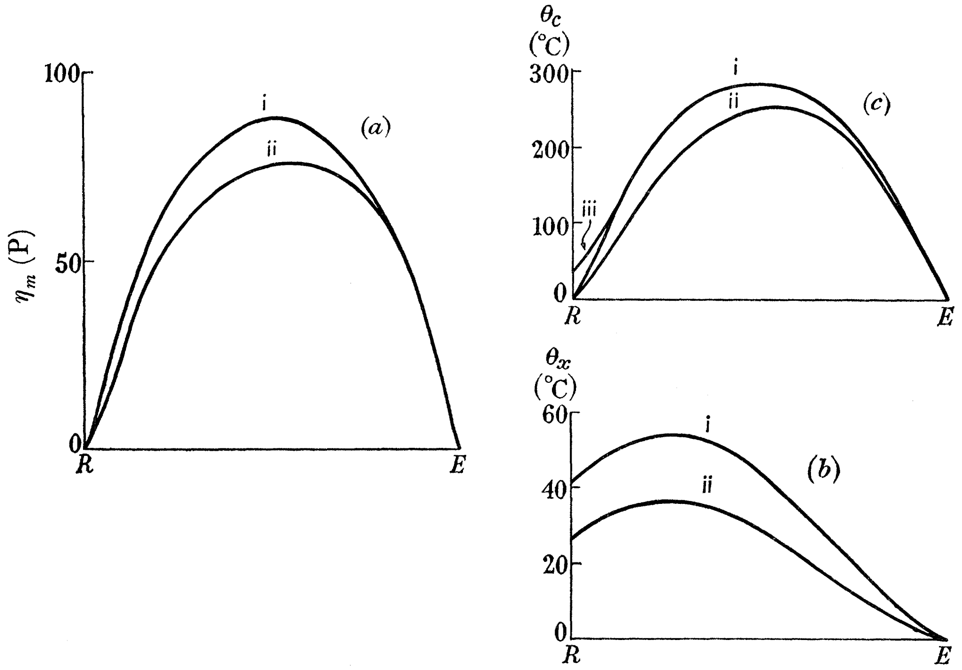
From figure 12 the value of T is $1\cdot02\times10^7\,\mathrm{dyn\,cm^{-1}}$. If (u_1+u_2) be taken as $1000\,\mathrm{cm\,s^{-1}}$ then for a slip of $400\,\mathrm{cm\,s^{-1}}$ u_1 is $300\,\mathrm{cm\,s^{-1}}$ and u_2 is $700\,\mathrm{cm\,s^{-1}}$. The values of θ_x given by equation (9·5) for the two disks are shown in figure 13 (b). These values were used to calculate mean values of $\exp(\gamma\theta_x)$ which were used to recalculate values of η_x (equation (9·1)) from which new values of ψ, $f(\psi)$ and η_m were obtained. These modified values of η_m are shown by curve (ii) of figure 13 (*a*). They do not differ greatly from those of curve (i) although the effect of a rise in surface temperature of 50 deg C is to reduce η_x by a factor approaching 3. This suggests that η_m is not very dependent upon η_x and this can be demonstrated analytically by differentiating equation (7·4) with respect to η_x. Then

$$\frac{\Delta\eta_m}{\eta_m} = \frac{1}{f(\psi)}\left[f(\psi)+\eta_x\frac{\partial}{\partial\eta_x}f(\psi)\right]\frac{\Delta\eta_x}{\eta_x}.$$

From the approximate expression for $f(\psi)$ (equation (7·5)) $\partial f(\psi)/\partial\eta_x$ may be obtained and it is then found that

$$\frac{\Delta\eta_m}{\eta_m} \approx \frac{1}{2\psi f(\psi)}\frac{\Delta\eta_x}{\eta_x}. \tag{9·6}$$

A plot of $2\psi f(\psi)$ is given in figure 14. At values of ψ of the order 100 such as occur in the present example the curve shows the proportional change in η_m to be one-sixth of the proportional change in η_x.



R = recess E = entry

FIGURE 13. The influence of the variations of surface temperature (θ_x) within the pressure zone. (*a*) η_m as a function of x: (i) variations of θ_x ignored; (ii) variations of θ_x considered. (*b*) θ_x as a function of x: (i) for disk with peripheral speed of 300 cm s^{-1}; (ii) for disk with peripheral speed of 700 cm s^{-1}. (*c*) θ_c as a function of x: (i) variations of θ_x ignored; (ii) variations of θ_x considered; (iii) $\theta_c+\bar{\theta}_x$ (coincident with curve (i) except at recess). Other data: load = 2×10^8 dyn cm^{-1}; $\bar{u}$ = 500 cm s^{-1}; u_2-u_1 = 400 cm s^{-1}.

The changes in θ_c, the excess temperature on the median plane, due to taking cognizance of the surface temperatures also are not large (cf. curves i and ii of figure 13 (*c*)). Perhaps, surprisingly, the effect is to diminish θ_c. But by definition θ_c is measured with respect to θ_x and when the mean value of θ_x at the two surfaces is added to θ_c (curve iii) it is seen that the oil temperatures on the median plane with respect to the surface temperature at entry are indistinguishable from the temperatures of curve i except at the extreme recess.

Curve ii of figure 13 (*a*) gives a value of $\bar{\eta}_m$ of 53 P compared with 60 P for curve (i), i.e. approximately 10 % less. Consequently, if in a further iterative cycle θ_x were calculated from curve (ii) values 10 % smaller would be obtained. This would increase η_x at most by 10 % and η_m would be increased by one-sixth of that at values of ψ of the order 100. Clearly further iterative cycles would not produce a materially different result.

Two further examples have been worked. In both $\bar{u}$ was taken as 500 cm s^{-1}. In one instance a high load (2×10^8 dyn cm^{-1}) was combined with a low speed of sliding (100 cm s^{-1}) whilst in the other a low load (1×10^8 dyn cm^{-1}) was combined with a high sliding speed (400 cm s^{-1}). Again in both instances it was found that $\bar{\eta}_m$ was reduced by approximately 10 %.

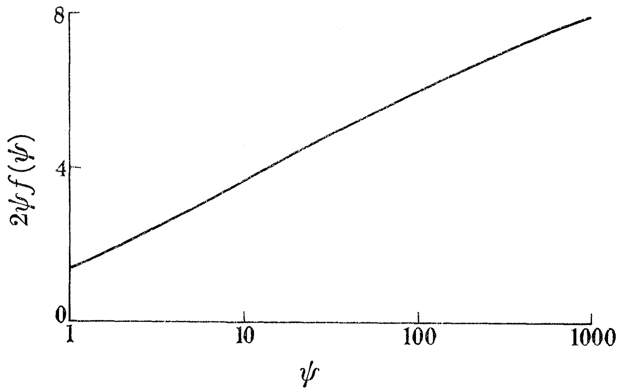


FIGURE 14. $2\psi f(\psi)$ as a function of ψ.

10. CONCLUSION

The generation and flow of heat within the oil film has been discussed and, for the range of conditions considered, it has been found that both in pure rolling and in rolling with sliding conduction of heat across the film to the surfaces of the disks is the most important mechanism of heat dissipation.

The temperature rise in the oil film has been assessed. In pure rolling it has been found that there is no temperature rise within the pressure zone; the temperature rise occurs on the entry side ahead of that zone. Furthermore, it has been proved that up to a $\bar{u}\eta_s$ of 150 dyn cm^{-1} the temperature rise which does occur is too small to account for the discrepancy between the relationships of film thickness to $\bar{u}\eta_s$ as found experimentally and as predicted by theory (part II). When sliding is introduced it has been found that the temperatures on the entry side remain small; a fact which explains the experimental observation that film thickness is insensitive to sliding when $\bar{u}\eta_s$ is held constant (part II). But the introduction of sliding does have a very marked influence upon the temperatures within the pressure zone. It has been shown that these temperatures depend upon pressure and hence upon load, but for constant η_s are independent of film thickness. At a sliding speed of 500 cm s^{-1} and a load of $1{\cdot}2 \times 10^8$ dyn cm^{-1} the maximum temperature rise of the oil has been assessed at 200 deg C (figure 5). Such a value brings into question whether the oil can pass through the pressure zone without chemical change.

An expression for the effective viscosity within the pressure zone has been developed. In the most severe example cited it has been found, for instance, that the introduction of 400 cm s^{-1} sliding causes the effective viscosity to fall in relation to its value in pure rolling by a factor of 50. Nevertheless, the temperature rise on the inlet side remains small and consequently film thickness is not greatly affected. (The experiments of part II showed a reduction of no more than 30 %.) It has also been shown that at high speeds of sliding the effective viscosity is largely independent of η_s. This fact carries the important implication that if an oil of higher viscosity (higher η_s) be used to afford to the surfaces greater protection in virtue of a thicker oil film, then there is little penalty to be paid by way of greater frictional heating and in fact at high speeds of sliding the frictional traction may be less with the thicker oil.

The tractions pass through a maximum as the sliding is increased. This implies that if the disks were used as a friction drive and the slip was allowed to exceed that at which the maximum traction occurs, then a demand for a greater output torque, which would lead to even greater sliding, would reduce the torque the drive can deliver. If the demand were maintained the output shaft would come to rest whilst the input shaft would accelerate; a situation of distressing consequence.

Lastly the influence of the rise in temperature at the surfaces of the disks, due to the heat flowing into them, upon the temperatures within the oil film and upon effective viscosity have been discussed. It has been shown that even when the rise in temperature at the surfaces is as high as 50 deg C the maximum temperature in the oil film is unchanged and that the effective viscosity is only 10 % less than it would be had the surfaces remained at constant temperature. The importance of this result is that in considering the experimental results such as those which will be presented in part IV their analysis need not be burdened in the first instance with the additional complication of considering variations of temperature at the surfaces of the disks.

In addition to the particular results which have been cited the analysis gives a knowledge of the temperatures and viscosities throughout the oil film and of the velocity profiles. It has been shown that as the sliding is increased slip within the film becomes concentrated upon the median plane where the temperature is highest, the cooler oil closer to the surfaces acting like a rigid deposit upon them. This is a situation akin to that postulated in boundary lubrication and suggests that a gradual transition between the states of hydrodynamic and boundary lubrication might exist.

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Appendix A. The power to overcome rolling friction in the presence of sliding

In calculating the oil temperatures in the presence of sliding (§ 5), in the expression taken for the rate of generation of heat per unit volume (equation (5·3)), it was assumed that the term in $\partial P/\partial x$ of equation (2·2) could be ignored. A consequence of neglecting that term is that the power required to overcome friction (W) is underestimated. The power implied by equation (5·3) will be termed W_s and the amount by which it is an underestimate will be termed W_R so

$$W = W_s + W_R.$$

From a consideration of the frictional traction W is obviously given by

$$W = \int_{x_1}^{x_2} \left[u_2 \left(\eta \frac{\partial u}{\partial y} \right)_{y=h} - u_1 \left(\eta \frac{\partial u}{\partial y} \right)_{y=0} \right] \mathrm{d}x$$

which, by equation (2·2) becomes

$$W = \int_{x_1}^{x_2} \left[\frac{\eta_m}{h} (u_2 - u_1)^2 - h \frac{\partial P}{\partial x} \left\{ \tfrac{1}{2}(u_1 + u_2) + (u_2 - u_1) \left(\frac{1}{2} - \frac{1}{m} \right) \right\} \right] \mathrm{d}x,$$

whereas W_s is given by integrating q (equation (5·3)) throughout the film and thereby it is found that

$$W_s = \int_{x_1}^{x_2} \frac{\eta_m (u_2 - u_1)^2}{h} \, \mathrm{d}x.$$

A comparison of W and W_s shows that

$$|W_R| < \left| -2 \int_{x_1}^{x_2} \frac{(u_1 + u_2)}{2} \, h \frac{\partial P}{\partial x} \, \mathrm{d}x \right|.$$

The expression on the right is twice the rolling friction. It has already been shown that the temperatures due to rolling friction are small (§ 4). Therefore the neglect of W_R cannot make the temperatures yielded by equation (5·10) seriously incorrect.

Appendix B. Dissipation by convection in the presence of sliding

In calculating the oil temperatures due to sliding (§ 5) conduction of heat to the surfaces of the disks was considered; convection was ignored. Here the effect of convection will be discussed.

If the temperatures of equation (5·10) did prevail then the heat which would be dissipated by convection on the median plane is given by

$$\rho c \bar{u} \, \partial \theta_c / \partial x$$

whilst the heat generated (equation (5·3)) is given by

$$q_c = \frac{\eta_m^2}{\eta_c} \frac{(u_2 - u_1)^2}{h_D^{*2}}.$$

Curves of q_c and of the heat which would be dissipated by convection are given in figure 15. In the calculations of the convection, values of $\partial\theta_c/\partial x$ were obtained from the temperature distributions of figure 5; $\bar{u}$ was taken as $500\,\mathrm{cm\,s^{-1}}$. In the calculations of q_c the values of η_m were taken from figure 10 and the values of h were obtained from figure 2 but an adjustment was made for the different rolling speed. (It was shown in part II that $h^*_D \propto (\bar{u}\eta_s)^{0\cdot5}$.)

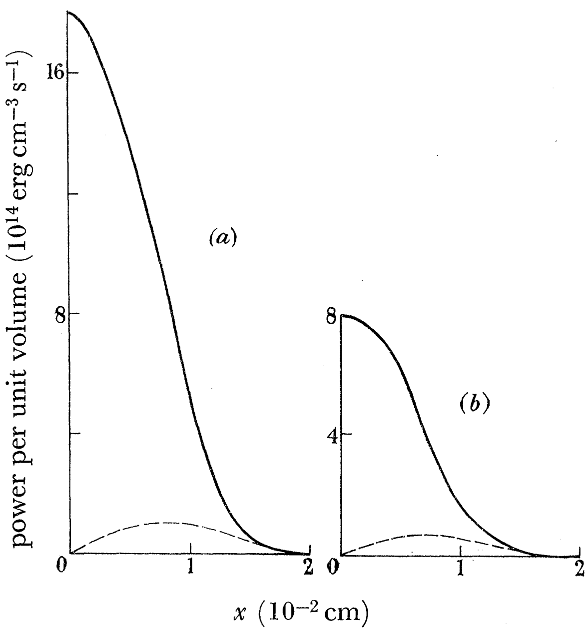

FIGURE 15. The power per unit volume on the median plane. ——, Generated (q_c); ----, dissipated by convection. (a) u_2-u_1, $200\,\mathrm{cm\,s^{-1}}$; (b) u_2-u_1, $100\,\mathrm{cm\,s^{-1}}$.

Figure 15 shows that in the examples cited, it is only at distances from the centre of the pressure zone greater than 1×10^{-2} cm that the dissipation by convection becomes comparable with q. At lesser distances the neglect of convection will clearly have little effect upon the calculated oil temperatures and therefore little effect upon the calculated values of η_m (figure 10). Considerable proportional errors in η_m, at distances greater than 1×10^{-2} cm, would not materially alter the average value of η_m and therefore would not materially alter the effective viscosity. Consequently with respect to the effective viscosity the dissipation by convection is unimportant.

As $\bar{u}$ increases, however, the ratio of the convection to q rises. From equation (5·10) the convection is given by

$$\rho c\bar{u}\frac{\partial\theta_c}{\partial x} = \frac{\rho c\bar{u}}{\gamma(\psi+1)}\frac{\partial\psi}{\partial x},$$

whereas in terms of ψ

$$q_c = \frac{8K\psi(\psi+1)\,(f(\psi))^2}{\gamma h^{*2}_D}.$$

Therefore the ratio of convection to q_c is

$$\frac{\rho c\bar{u}\,\partial\theta_c/\partial x}{q_c} = \frac{\rho c\bar{u}h^{*2}_D}{8K\psi(\psi+1)^2\,(f(\psi))^2}\frac{\partial\psi}{\partial x}.$$

Because $h^*_D \propto (\bar{u})^{0\cdot5}$ when η_s is constant it follows that the ratio will then be proportional to $(\bar{u})^2$. Therefore as the rolling speed increases, the errors due to neglect of convection become more serious. But even so, over the complete pressure zone the errors will be self-balancing to some degree. If convection were taken into account its influence would be to depress the temperatures while they were rising in the direction of the oil flow and the converse. This would make the temperature distributions asymmetrical but would not greatly influence the average value of η_m.

Appendix C. The values of m

When $y = h/m$, $\eta = \eta_m$ and when η_c and η_m have been expressed in terms of ψ it follows from equation (5·13) that

$$(1-2/m)\sqrt{\{\psi(\psi+1)\}}f(\psi) = \ln[\sqrt{\{(\psi+1)f(\psi)-1\}}+\sqrt{\{(\psi+1)f(\psi)\}}]. \quad \text{(C 1)}$$

This shows that m is solely a function of ψ (figure 16). If the sign of one side of the above expression be changed the new value of m, which is then obtained, and which will be denoted by n, satisfies the expression

$$\frac{1}{n} = 1 - \frac{1}{m}.$$

Clearly n corresponds to the other position where $(\partial u/\partial y) = (u_2 - u_1)/h$ (figure 8).

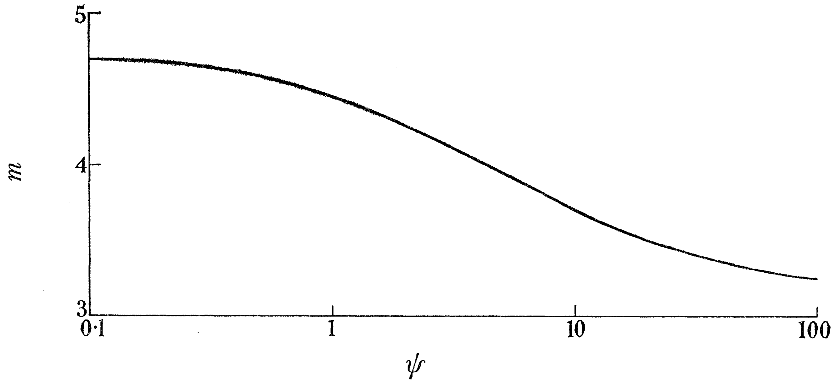

Figure 16. m as a function of ψ.

For a linear profile m is indeterminate but the slightest change from linearity produces discrete places where $(\partial u/\partial y) = (u_2 - u_1)/h$ and by expanding equation (C 1) in powers of ψ it is found that

$$\lim_{\psi\to 0} m = 2\sqrt{3}/(\sqrt{3}-1) = 4·73.$$

As ψ increases (greater sliding) the velocity profile changes towards a configuration with a step at $y = \frac{1}{2}h$. The places where $\partial u/\partial y = (u_2 - u_1)/h$ move towards each other and in the limit coincide with $y = \frac{1}{2}h$. Indeed from equation (C 1) it may be seen that

$$\lim_{\psi\to\infty} m = 2.$$